n_3 of a third, and so on to n_k of a kth kind, where

$$n_1 + n_2 + n_3 + \cdots + n_k = n.$$

Then the number of distinguishable arrangements of all n is

$$\frac{n!}{n_1!\, n_2!\, n_3! \cdots n_k!}. \tag{7}$$

Example 3. How many distinguishable permutations are there of the letters $B\,A\,R\,B\,A\,R\,I\,A\,N$?

Solution: We have $n = 9$ letters, of which $n_1 = 2$ are B's, $n_2 = 3$ are A's, $n_3 = 2$ are R's, $n_4 = 1$ is an I, and $n_5 = 1$ is an N. Hence, by (7), the number of distinguishable permutations is

$$\frac{9!}{2!\,3!\,2!\,1!\,1!} = 15{,}120.$$

Exercises [A]

1. Evaluate. (a) $P(6, 3)$ (b) $P(50, 2)$ (c) $P(100, 1)$
2. Solve for n. (a) $P(n, 2) = 12$
   (b) $2P(n, 3) = 3P(n - 1, 3)$
   (c) $P(n, 4) = 6P(n, 2)$
3. We have not defined $P(n, 0)$. What value does (6) suggest for $P(n, 0)$?
4. A game requires laying out a deck of cards in 4 rows of 13 cards each. How many layouts are possible?
5. There are 20 men on a baseball squad. How many batting orders of 9 men are possible? How many if 4 particular players must always bat in a set order?
6. (a) How many 5-letter words can be written with the English alphabet if repetitions of letters are allowed?
   (b) If all these words were written, in how many would repetitions actually occur?
   (c) How many 5-letter words can be written if repetitions are permitted but double letters are not?
7. How many distinguishable permutations are there of the following letters?
   (a) $S\,U\,S\,U\,R\,R\,U\,S$
   (b) $A\,B\,R\,A\,C\,A\,D\,A\,B\,R\,A$

8. (a) In how many ways can Tom, Dick, Harry, Joe, and Sam form a line?
   (b) In how many of these is Tom not ahead of Dick?
   (c) In how many is Tom directly behind Dick?
9. In how many orders can 4 red blocks and 6 blue blocks be lined up in a row?
10. In how many orders can the letters $F\,A\,C\,E\,T\,I\,O\,U\,S$ be written without having two vowels come together?
11. In how many positive integers less than 1,000,000 is no digit repeated?
12. (a) In how many orders can the letters $S\,T\,A\,T\,I\,S\,T\,I\,C\,S$ be arranged?
    (b) How many of these begin and end with S?
    (c) In how many do the two I's come together?
13. How many distinguishable ways are there of arranging 6 identical brown eggs and 6 identical white eggs in a 12-egg carton?
14. (a) In how many ways can 10 children form a line?
    (b) In how many ways can they form a circle, if only their relative positions are considered?
    (c) In how many ways can n children form a circle?
15. In how many ways can 5 men and 5 women form a circle so that no two women are side by side, if only their relative positions are considered?
16. In how many orders can 5 beads of different colors be arranged on a circular wire?
17. In how many different orders can one walk 5 blocks east and 3 blocks south?
18. Each permutation of the digits 1, 1, 1, 2, 2, 3 represents a six-digit number. If these numbers are listed in order of increasing size,
    (a) what number is fifteenth in the list?
    (b) how far down the list is 322,111?

Exercises [B]

1. (a) How many permutations of 5 S's and 2 D's can be distinguished?
   (b) A subset of 5 elements is to be selected from a set of 7 elements by the following procedure: each element of the original set is to be labeled S or D (but not both), and the elements labeled S are to be selected for the subset, while the elements labeled D are to be re-

a term $x^{n-r}y^r$ of the product arises from choosing the x in each of $n-r$ factors and the y in the other r factors. Since this can be done in $C(n, n-r) = C(n, r)$ ways, the simplified product will contain the term $C(n, r)x^{n-r}y^r$. We are able to conclude, independently of the derivation on page 452, that

$$(x+y)^n = \sum_{r=0}^{n} C(n, r)x^{n-r}y^r.$$

We now have two notations, $C(n, r)$ and $\binom{n}{r}$, for the same expression. In keeping with contemporary usage, we shall generally (though not invariably) prefer the latter.

The relationship between the binomial coefficient $\binom{n}{r}$ and the number of combinations $C(n, r)$ permits a proof of Pascal's Law that is rather more elegant than the one on page 451. In terms of combinations, rather than binomial coefficients, this law becomes the following.

▶ **Theorem.**

$$C(n, r-1) + C(n, r) = C(n+1, r).$$

Proof: Let E be a set containing $n+1$ elements. The number of r-element subsets of E is, by definition, $C(n+1, r)$. Now consider some fixed element of E; call it x. Each r-element subset of E either contains x or does not contain x. The number of those that do contain x is $C(n, r-1)$, because including x leaves n elements of E from which to choose the remaining $r-1$ elements of each subset. The number of those that do not contain x is $C(n, r)$, because excluding x leaves n elements of E from which to choose all r elements of each subset. Hence, by (*1*),

$$C(n, r-1) + C(n, r) = C(n+1, r).$$

Exercises [A]

1. Compute the number of 2-card subsets of a deck of 52 cards.
2. (a) A man has 12 friends whom he would like to invite to his home, but he has only room enough for 9 guests. In how many ways can he select the 9? (b) If he had only room enough for 3 guests, in how many ways could he select the 3?
3. Eighteen boys gather to play baseball. In how many ways can they be divided into 2 teams, 9 boys to a team?

4. Each of the 8 teams in a bowling league bowls against each of the others 4 times. How many matches are bowled?

5. (a) How many 3-element subsets can be chosen from $\{A, B, C, D, E, F\}$?
   (b) How many of these include A?
   (c) How many include A but not B?
   (d) How many include A and B?
   (e) How many exclude A and B?
   (f) How many include A or B?

6. A collector has ten antique coins. In how many ways can he select two or more to sell?

7. (a) How many committees of 6 people can be chosen from a group of 5 men and 6 women?
   (b) How many of these committees will consist of 3 men and 3 women?
   (c) Once a committee has been determined, how many ways are there of choosing a chairman and a secretary?

8. (a) How many lines are determined by n points, no three of them collinear?
   (b) How many diagonals has a polygon of n sides?

9. What is the maximum number of intersections of five lines?

10. (a) If no four of five given points are coplanar, how many lines do they determine?
    (b) How many planes do they determine?

11. If four lines are coplanar, no two of them parallel and no three concurrent, how many triangles do they form?

12. (a) If 16 points are marked on a circle, how many triangles have their vertices at these points?
    (b) If 16 points are marked on a square, 4 points on each side, how many triangles have their vertices at these points?

13. A "deck" of 12 cards is made up of the kings, queens, and jacks from an ordinary deck. (a) How many 3-card hands can be selected from this deck? (b) How many of these hands contain (i) 3 of a kind (3 cards of the same denomination)? (ii) 3 cards of the same suit? (iii) a sequence (J-Q-K)? (iv) a pair, but not 3 of a kind? (v) no pair?

14. A store stocks 4 kinds of candy bars. In how many ways can 3 candy bars be chosen?

Pages 476–477

1. 23 3. 271 5. 271 7. (a) 16 (b) 12 9. 265 11. 8190 13. 59
15. 104

Pages 483–484

1. (a) A (b) B 3. $\emptyset$ 5. 7
7. (a) 1, 2, 3, 4, 5, 6 (b) 1, 4 (c) 1, 2, 3, 5, 6, 8 (d) 5, 8

Pages 488–489

1. (a) 120 (b) 2450 (c) 100 3. 1 5. $\frac{20!}{11!}$ 7. (a) 560 (b) 83160 9. 210
11. 5,652,770 13. 924 15. 2880 17. 56

Pages 489–490

1. (a) 21 (b) 21 (c) 21 3. (a) 120 (b) 120

Pages 494–496

1. 1326 3. 48620 5. (a) 20 (b) 10 (c) 6 (d) 4 (e) 4 (f) 16
7. (a) 462 (b) 200 (c) 30 9. 10 11. 4
13. (a) 220 (b) (i) 12 (ii) 4 (iii) 64 (iv) 144 (v) 64 15. 1296

Pages 496–497

1. 13,824 3. (a) 40 (b) 624 (c) 3744 (d) 5108 (e) 10,200 (f) 54,912
(g) 123,552 (h) 1,098,240 5. (a) 120 (b) 90 (c) 6

Chapter Review, Page 497

1. 1120 3. 1536

Pages 501–502

1. (a) (b) (f) 3. (a) $\{(1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2)\}$
(b) (i) $\{(2, 1), (3, 1), (3, 2)\}$ (ii) $\{(3, 1), (3, 2)\}$ (iii) $\{(1, 2), (3, 2)\}$
(iv) $\{(1, 3), (3, 1)\}$ (v) $\{(2, 1), (3, 1), (3, 2)\}$ (vi) $\{(3, 1), (3, 2)\}$
(vii) $\{(1, 2), (1, 3), (3, 1), (3, 2)\}$ (viii) $\emptyset$ (ix) $\{(3, 2)\}$ (x) $\{(3, 2)\}$

Pages 508–509

9. $P(E) = \frac{1}{4}$, $P(\overline{E}) = \frac{3}{4}$ 13. (a) $\frac{1}{221}$ (b) $\frac{1}{17}$ (c) $\frac{1}{17}$ (d) $\frac{19}{34}$ (e) $\frac{33}{221}$
(f) $\frac{16}{17}$ (g) $\frac{29}{442}$ (h) $\frac{116}{221}$